

# A New Proof of Global Existence for the Dirac Klein–Gordon Equations in One Space Dimension

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## 1. INTRODUCTION

In this paper we study the Dirac and Klein–Gordon equations, coupled through the Yakawa interaction, in one space dimension. They read as

$$\mathcal{D}\psi = g\phi\psi - m\psi \quad (1)$$

$$\square\phi = g\bar{\psi}\psi. \quad (2)$$

Here  $\phi(t, x)$  is a real valued scalar field and  $\psi(t, x)$  is a 2-spinor field. The Dirac operator  $\mathcal{D}$  is defined by  $\mathcal{D} = -i\gamma^\mu \partial_\mu$  where  $\gamma^\mu$ ,  $\mu = 0, 1$  are the Dirac matrices. The wave operator  $\square$  is defined by  $\square = -\partial_t^2 + \partial_x^2$ . Then  $\mathcal{D}^2 = \square$ . We denote by  $\psi^\dagger$  the complex conjugate transpose of  $\psi$  and we define  $\bar{\psi} = \psi^\dagger \gamma^0$ . We prescribe initial data at time  $t = 0$ ,

$$\psi(0, x) = g(x), \quad \phi(0, x) = \phi_0(x), \quad \partial_t \phi(0, x) = \phi_1(x). \quad (3)$$

The Cauchy problem (1)–(3) was first studied by Chadam in [1]. He proves that if  $g \in H^1$ ,  $\phi_0 \in H^1$ ,  $\phi_1 \in L^2$  then a unique global solution exists. Chadam's proof has two steps. In the first step he proves that the Cauchy problem (1)–(3) is locally well posed in the space indicated by the above

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choice of initial data. In the second step, using a clever bootstrap argument, he proves that the corresponding norms of  $\phi$  and  $\psi$  never blow up. This implies global existence.

In this paper we shall give a simpler proof using the following technique of S. Klainerman. In order to prove global existence one first proves local existence of energy class solutions. Since the energy is conserved this automatically implies global existence. This method was used in [3] to prove global existence for the Maxwell Klein–Gordon equations and in [4] to prove global existence for the Yang–Mills equations. In our case the energy is given by the formula

$$\int_{\mathbb{R}} \{e(\phi)(t, x) + \psi^\dagger(t, x) \gamma^1 \gamma^0 \partial_x \psi(t, x) - i(M - g\phi(t, x)) \psi^\dagger(t, x) \gamma^0 \psi(t, x)\} dx,$$

where  $e(\phi) = \frac{1}{2} |\partial_t \phi|^2 + \frac{1}{2} |\partial_x \phi|^2 + m^2 \phi^2$ .

Solutions with  $\psi(t, \cdot) \in H^{1/2}(\mathbb{R})$ ,  $\phi(t, \cdot) \in H^1(\mathbb{R})$  are called energy class solutions. In order to prove local existence in the energy class we would have to improve Chadam's local existence result by  $\frac{1}{2}$  derivative in  $\psi$ . However, the energy density is not positive! Therefore conservation of energy does not imply global existence of energy class solutions. The only positive conserved quantity is the charge  $\int |\psi(t, x)|^2 dx$ . Because of this we shall have to prove local existence of "charge class" solutions, i.e., solutions with  $\psi(t, \cdot) \in L^2(\mathbb{R})$ .

Our proof will have two steps. In the first step we shall prove that the Cauchy problem (1)–(3) is locally well posed if  $g \in L^2$ ,  $\phi_0 \in H^1$ ,  $\phi_1 \in L^2$ , thus improving Chadam's local existence theorem by one derivative in  $\psi$ . This improvement is due to the fact that the right hand side of (2) satisfies Klainerman's null condition (in a sense to be made precise later). A null-form estimate will be employed that will allow us to estimate  $\phi$  in  $H^1$  even though  $\psi$  is not assumed to have any derivatives in  $L^2$ . The proof of this null-form estimate is elementary in one space dimension. In the second step we shall prove the global existence of the solution constructed in the first step. Here our choice of spaces for the local problem comes to our aid. In the spirit of [3, 4] we use the basic conservation law of (1)–(2), namely conservation of charge, to guarantee that  $\psi$  never blows up in  $L^2$ . Using one more time our null-form estimate we then show that  $\phi$  doesn't blow up in  $H^1$ , thus completing the proof of global existence.

This line of proof is possible thanks to the null structure of the nonlinear terms of our system. Actually our local existence result can be further improved using more elaborate space time estimates for null forms (see the Remarks at the end of the paper).

Our main result will be the following:

**THEOREM 1.** *The Cauchy problem for the Dirac Klein-Gordon equations in one space dimension with data*

$$\psi(0, \cdot) \in L^2, \quad \phi(0, \cdot) \in H^1, \quad \partial_t \phi(0, \cdot) \in L^2 \quad (4)$$

*admits a unique global solution in the space*

$$(\psi, \phi) \in C([0, +\infty), L^2) \times (C^1([0, +\infty), H^1) \cap C^0([0, +\infty, L^2))). \quad (5)$$

## 2. THE LOCAL PROBLEM

### 2.1. Linear Estimates

In this section we present the linear estimates for the Dirac equation and the wave equation that will be used in the sequel.

For any space time functions  $u(t, x)$ ,  $v(t, x)$  we define Klainerman's null forms as

$$Q_0(u, v) = \partial_t u \partial_t v - \partial_x u \partial_x v \quad (6)$$

$$Q_1(u, v) = \partial_t u \partial_x v - \partial_x u \partial_t v. \quad (7)$$

We then have the following space-time estimate:

**LEMMA 1.** *Let  $u$ ,  $v$  be the solutions to the following initial value problems,*

$$\square u = F, \quad u(0, \cdot) = f_0, \quad \partial_t u(0, \cdot) = f_1$$

$$\square v = G, \quad v(0, \cdot) = g_0, \quad \partial_t v(0, \cdot) = g_1$$

*and let  $Q$  be any of the null forms defined above. Then*

$$\begin{aligned} \|Q(u, v)\|_{L^2(\mathbb{R} \times \mathbb{R})} &\leq c \left( \|f_0\|_{H^1(\mathbb{R})} + \|f_1\|_{L^2(\mathbb{R})} + \int_0^T \|F(t, \cdot)\|_{L^2(\mathbb{R})} dt \right) \\ &\quad \times \left( \|g_0\|_{H^1(\mathbb{R})} + \|g_1\|_{L^2(\mathbb{R})} + \int_0^T \|G(t, \cdot)\|_{L^2(\mathbb{R})} dt \right). \end{aligned}$$

*Proof.* This estimate is the analogue of Theorem 2 of [2] for the case of dimension one. The proof is immediate from the explicit form of the solutions. Thanks to Duhamel's principle it suffices to prove it in the special case  $F = f_1 = 0$  and  $G = g_1 = 0$ . Then  $u$  and  $v$  are given by

$$u(t, x) = \frac{1}{2} (f_0(x+t) + f_0(x-t))$$

$$v(t, x) = \frac{1}{2} (g_0(x+t) + g_0(x-t)).$$

Therefore

$$Q_0(u, v)(t, x) = -\frac{1}{2} (f'_0(x-t) g'_0(x+t) + f'_0(x+t) g'_0(x-t)).$$

Hence

$$\begin{aligned} \|Q_0(u, v)\|_{L^2(\mathbb{R} \times \mathbb{R})} &\leq c \left( \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f'_0(x-t) g'_0(x+t)|^2 dx dt \right. \\ &\quad \left. + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f'_0(x+t) g'_0(x-t)|^2 dx dt \right)^{1/2} \\ &\leq c \left( \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f'_0(\xi) g'_0(\eta)|^2 d\xi d\eta \right)^{1/2} \\ &\leq c \|f_0\|_{H^1(\mathbb{R})} \|g_0\|_{H^1(\mathbb{R})}. \end{aligned}$$

The proof for  $Q_1$  is similar. ■

LEMMA 2. *Let  $\psi(t, x)$  be the solution of*

$$\mathcal{D}\psi = G, \quad \psi(0, x) = g(x),$$

where  $g \in L^2(\mathbb{R})$ ,  $G \in L^1_{loc}(\mathbb{R} \rightarrow L^2(\mathbb{R}))$ . Then, for any  $T > 0$  and any  $t \in [0, T]$  we have

$$\|\psi(t, \cdot)\|_{L^2(\mathbb{R})} \leq c \left( \|g\|_{L^2(\mathbb{R})} + \int_0^T \|G(s, \cdot)\|_{L^2(\mathbb{R})} ds \right).$$

*Proof.* In the special case  $G = 0$  we have

$$\|\psi(t, \cdot)\|_{L^2(\mathbb{R})} = \|g\|_{L^2(\mathbb{R})}.$$

This is the law of conservation of charge for the linear Dirac equation. The general case follows from this special one and Duhamel's principle. ■

LEMMA 3. *let  $\psi$  be a 2-spinor field on  $\mathbb{R} \times \mathbb{R}$ . Let  $\Psi$  be the solution of*

$$\square \Psi = \mathcal{D}\psi, \quad \Psi(0, \cdot) = 0, \quad \partial_t \Psi(0, \cdot) = i\gamma^0 \psi(0, \cdot).$$

Then

$$\begin{aligned} \mathcal{D}\Psi &= \psi \\ -\bar{\psi}\psi &= Q_1(\Psi^\dagger, \gamma^1 \Psi) + Q_0(\Psi^\dagger, \gamma^0 \Psi). \end{aligned}$$

*Proof.* Observe that  $\mathcal{D}\Psi - \psi$  satisfies

$$\mathcal{D}(\mathcal{D}\Psi - \psi) = 0, \quad (\mathcal{D}\Psi - \psi)(0, \cdot) = 0.$$

Hence  $\mathcal{D}\Psi - \psi = 0$ . Recall now that the Dirac matrices satisfy  $(\gamma^0)^2 = I$ ,  $(\gamma^1)^2 = -I$ ,  $\gamma^0\gamma^1 + \gamma^1\gamma^0 = 0$ ,  $(\gamma^0)^* = \gamma^0$ ,  $(\gamma^1)^* = -\gamma^1$ . Therefore

$$\begin{aligned} \bar{\psi} &= \overline{\mathcal{D}\Psi} \\ &= i(\partial_t \Psi^\dagger (\gamma^0)^* + \partial_x \Psi^\dagger (\gamma^1)^*) \gamma^0 \\ &= i(\partial_t \Psi^\dagger - \partial_x \Psi^\dagger \gamma^1 \gamma^0). \end{aligned}$$

Hence

$$\begin{aligned} -\bar{\psi}\psi &= (\partial_t \Psi^\dagger - \partial_x \Psi^\dagger \gamma^1 \gamma^0)(\gamma^0 \partial_t \Psi + \gamma^1 \partial_x \Psi) \\ &= \partial_t \Psi^\dagger \partial_x (\gamma^1 \Psi) - \partial_x \Psi^\dagger \partial_t (\gamma^1 \Psi) + \partial_t \Psi^\dagger \partial_t (\gamma^0 \Psi) - \partial_x \Psi^\dagger \partial_x (\gamma^0 \Psi) \\ &= \mathcal{Q}_1(\Psi^\dagger, \gamma^1 \Psi) + \mathcal{Q}_0(\Psi^\dagger, \gamma^0 \Psi). \quad \blacksquare \end{aligned}$$

## 2.2. Local Existence

**DEFINITION.** A charge class solution of (1)–(2) is a pair  $(\psi, \phi)$  defined on some slab  $[0, T) \times \mathbb{R}$  such that the equations are satisfied in the sense of distributions and

$$(\psi, \phi) \in C([0, T), L^2) \times (C^1([0, T), H^1) \cap C^0([0, T), L^2)).$$

In this section we shall prove the following local existence theorem:

**THEOREM 2.** *Let*

$$g \in L^2(\mathbb{R}), \quad \phi_0 \in H^1(\mathbb{R}), \quad \phi_1 \in L^2(\mathbb{R}).$$

*Define*

$$J(0) = \|g\|_{L^2(\mathbb{R})} + \|\phi_0\|_{H^1(\mathbb{R})} + \|\phi_1\|_{L^2(\mathbb{R})}.$$

*Then there exist a  $T > 0$ , depending only on  $J(0)$ , and a unique charge class solution of (1)–(3) defined on  $[0, T) \times \mathbb{R}$ .*

Theorem 2 follows by standard arguments from the following proposition (see, for example, Sections 3 and 4 of [2]).

**PROPOSITION 1.** *Let  $T > 0$  and  $(\psi, \phi)$ ,  $(\psi', \phi')$  be two charge class solutions of (1)–(2). Define*

$$\begin{aligned}
J(0) &= \|\psi(0, \cdot)\|_{L^2(\mathbb{R})} + \|\phi(0, \cdot)\|_{H^1(\mathbb{R})} + \|\partial_t \phi(0, \cdot)\|_{L^2(\mathbb{R})} \\
J'(0) &= \|\psi'(0, \cdot)\|_{L^2(\mathbb{R})} + \|\phi'(0, \cdot)\|_{H^1(\mathbb{R})} + \|\partial_t \phi'(0, \cdot)\|_{L^2(\mathbb{R})} \\
\Delta(T) &= \sup_{0 \leq t \leq T} (\|\psi(t, \cdot) - \psi'(t, \cdot)\|_{L^2(\mathbb{R})} + \|\phi(t, \cdot) - \phi'(t, \cdot)\|_{H^1(\mathbb{R})} \\
&\quad + \|\partial_t \phi(t, \cdot) - \partial_t \phi'(t, \cdot)\|_{L^2(\mathbb{R})}).
\end{aligned}$$

Then there exist  $\varepsilon > 0$ ,  $C > 0$ , depending only on  $J(0)$ ,  $J'(0)$  such that if  $T < \varepsilon$  then  $\Delta(T) \leq C\Delta(0)$ .

To prove Proposition 1 we shall need the following Lemma which will also be used in the proof of global existence. Note that no smallness assumption on  $T$  is made here.

**LEMMA 4.** *Let  $T > 0$  and let  $(\psi, \phi)$  be a charge class solution of (1)–(2). Define*

$$J(T) = \sup_{0 \leq t \leq T} (\|\psi(t, \cdot)\|_{L^2(\mathbb{R})} + \|\phi(t, \cdot)\|_{H^1(\mathbb{R})} + \|\partial_t \phi(t, \cdot)\|_{L^2(\mathbb{R})}).$$

Then there exists a constant  $C > 0$ , depending only on  $T$  and  $J(0)$ , such that  $J(T) \leq C = C(T, J(0))$ .

*Proof.* Thanks to conservation of charge for the nonlinear system (1)–(2) we have

$$\|\psi(\cdot, \cdot)\|_{L^2(\mathbb{R})} = \|\psi(0, \cdot)\|_{L^2(\mathbb{R})}. \quad (8)$$

For classical solutions this conservation law is a consequence of the invariance under the one parameter groups of transformations

$$\psi \rightarrow e^{i\alpha} \psi, \quad \bar{\psi} \rightarrow e^{-i\alpha} \bar{\psi}$$

and Noether's principle. For charge class solutions one first approximates the initial data  $g, \phi_0, \phi_1$  by smooth compactly supported data  $g_n, \phi_{0,n}, \phi_{1,n}$ , and then passes to the limit as  $n \rightarrow \infty$  in the conservation of charge for the corresponding solutions  $\psi_n, \phi_n$ .

This takes care of the first term in  $J(T)$ . We now estimate  $\phi$ . We shall first show that

$$\|\phi(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq C(T, J(0)). \quad (9)$$

Write  $\phi = \phi_L + \phi_N$  where  $\phi_L$  is the solution of

$$\square \phi_L = 0, \quad \phi_L(0, \cdot) = \phi(0, \cdot), \quad \partial_t \phi_L(0, \cdot) = \partial_t \phi(0, \cdot)$$

and  $\phi_N$  is the solution of

$$\square \phi_N = \bar{\psi} \psi, \quad \phi_N(0, \cdot) = 0, \quad \partial_t \phi_N(0, \cdot) = 0.$$

Apply the standard energy estimate and the one dimensional Sobolev inequality to get

$$\begin{aligned}
 \|\phi_L(t, \cdot)\|_{L^\infty(\mathbb{R})} &\leq c \|\phi_L(t, \cdot)\|_{H^1(\mathbb{R})} \\
 &\leq c(T)(\|\phi(0, \cdot)\|_{H^1(\mathbb{R})} + \|\partial_t \phi(0, \cdot)\|_{L^2(\mathbb{R})}) \\
 &\leq c(T) J(0).
 \end{aligned} \tag{10}$$

To estimate  $\phi_N$  we use the explicit form of the solution and the law of conservation of charge (8),

$$\begin{aligned}
 |\phi_N(t, x)| &\leq c \left| \int_0^t \int_{x-t+s}^{x+t-s} \bar{\psi}(s, y) \psi(s, y) dy ds \right| \\
 &\leq c \int_0^T \int_{-\infty}^{+\infty} |\psi(s, y)|^2 dy ds \\
 &\leq c \int_0^T \|\psi(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\
 &\leq cT \|\psi(0, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\
 &\leq cTJ(0).
 \end{aligned} \tag{11}$$

Estimate (9) is then a consequence of estimates (10) and (11). We now estimate the two terms of  $J(T)$  involving  $\phi$ . Apply the standard energy estimate to get

$$\|\phi(t, \cdot)\|_{H^1(\mathbb{R})} + \|\partial_t \phi(t, \cdot)\|_{L^2(\mathbb{R})} \leq c(T) \left( J(0) + \int_0^T \|\square \phi(s, \cdot)\|_{L^2(\mathbb{R})} ds \right). \tag{12}$$

Define  $\Psi$  as in Lemma 3. Use Lemmas 3 and 2 to get

$$\begin{aligned}
 &\int_0^T \|\square \phi(s, \cdot)\|_{L^2(\mathbb{R})} ds \\
 &\leq \int_0^T \|\overline{\psi(s, \cdot)} \psi(s, \cdot)\|_{L^2(\mathbb{R})} ds \\
 &\leq \int_0^T \|\mathcal{Q}_1(\Psi^\dagger, \gamma^1 \Psi) + \mathcal{Q}_0(\Psi^\dagger, \gamma^0 \Psi)\|_{L^2(\mathbb{R})} ds
 \end{aligned}$$

$$\begin{aligned}
&\leq cT^{1/2} \|Q_1(\Psi^\dagger, \gamma^1 \Psi') + Q_0(\Psi^\dagger, \gamma^0 \Psi)\|_{L^2(\mathbb{R} \times \mathbb{R})} \\
&\leq cT^{1/2} \left( \|\partial_t \Psi(0, \cdot)\|_{L^2(\mathbb{R})} + \int_0^T \|\square \Psi(s, \cdot)\|_{L^2(\mathbb{R})} ds \right)^2 \\
&\leq cT^{1/2} \left( J(0) + \int_0^T \|\phi(s, \cdot) \psi(s, \cdot) - m\psi(s, \cdot)\|_{L^2(\mathbb{R})} ds \right)^2 \\
&\leq cT^{1/2} \left( J(0) + \int_0^T \|\phi(s, \cdot)\|_{L^\infty(\mathbb{R})} \|\psi(s, \cdot)\|_{L^2(\mathbb{R})} ds + TmJ(0) \right)^2 \\
&\leq C(T, J(0)), \tag{13}
\end{aligned}$$

where we have used (8) and (9) to get the last inequality. This completes the proof of the lemma. ■

*Proof of Proposition 1.* Let  $J(T)$  be defined as in Lemma 4 and let  $J'(T)$  be the corresponding quantity for the primed solution. Let  $\psi$  be defined as in Lemma 3 and let  $\Psi'$  be the corresponding 2-spinor for  $\psi'$ . Then

$$\mathcal{D}(\psi - \psi') = g(\phi - \phi') \psi + g\phi'(\psi - \psi') - m(\psi - \psi') \tag{14}$$

$$\begin{aligned}
\square(\phi - \phi') &= Q_1(\Psi^\dagger - \Psi'^\dagger, \gamma^1 \Psi') + Q_1(\Psi^\dagger, \gamma^1 \Psi' - \gamma^1 \Psi) \\
&\quad Q_0(\Psi'^\dagger - \Psi^\dagger, \gamma^0 \Psi') + Q_0(\Psi^\dagger, \gamma^0 \Psi' - \gamma^0 \Psi). \tag{15}
\end{aligned}$$

Apply Lemma 2 to (14) to get

$$\|\psi(t, \cdot) - \psi'(t, \cdot)\|_{L^2(\mathbb{R})} \leq c \left( A(0) + \int_0^T \|\mathcal{D}(\psi - \psi')\|_{L^2(\mathbb{R})} ds \right). \tag{16}$$

In view of (14) we have

$$\begin{aligned}
\int_0^T \|\mathcal{D}(\psi - \psi')\|_{L^2(\mathbb{R})} ds &\leq \int_0^T \|(\phi(s, \cdot) - \phi'(s, \cdot)) \psi(s, \cdot)\|_{L^2(\mathbb{R})} ds \\
&\quad + \int_0^T \|\phi'(s, \cdot)(\psi(s, \cdot) - \psi'(s, \cdot))\|_{L^2(\mathbb{R})} ds \\
&\quad + m \int_0^T \|\psi(s, \cdot) - \psi'(s, \cdot)\|_{L^2(\mathbb{R})} ds. \tag{17}
\end{aligned}$$



For the first term on the right hand side of (17) we have the estimate

$$\begin{aligned}
 & \int_0^T \|(\phi(s, \cdot) - \phi'(s, \cdot)) \psi(s, \cdot)\|_{L^2(\mathbb{R})} ds \\
 & \leq \int_0^T \|\phi(s, \cdot) - \phi'(s, \cdot)\|_{L^\infty(\mathbb{R})} \|\psi(s, \cdot)\|_{L^2(\mathbb{R})} ds \\
 & \leq c \int_0^T \|\phi(s, \cdot) - \phi'(s, \cdot)\|_{H^1(\mathbb{R})} \|\psi(s, \cdot)\|_{L^2(\mathbb{R})} ds \\
 & \leq cTJ(0) \Delta(T).
 \end{aligned} \tag{18}$$

Similarly

$$\int_0^T \|\phi'(s, \cdot)(\psi(s, \cdot) - \psi'(s, \cdot))\|_{L^2(\mathbb{R})} ds \leq cTJ(0) \Delta(T). \tag{19}$$

From (18)–(19) we get

$$\int_0^T \|\mathcal{D}(\psi(s, \cdot) - \psi'(s, \cdot))\|_{L^2(\mathbb{R})} ds \leq c(1 + J(0)) T \Delta(T). \tag{20}$$

From (20) and (16) we get

$$\|\psi(t, \cdot) - \psi'(t, \cdot)\|_{L^2(\mathbb{R})} \leq c(\Delta(0) + c(1 + J(0)) T \Delta(T)). \tag{21}$$

Apply the standard energy estimate to (15) to get

$$\begin{aligned}
 & \|\phi(t, \cdot) - \phi'(t, \cdot)\|_{H^1(\mathbb{R})} + \|\partial_t \phi(t, \cdot) - \partial_t \phi'(t, \cdot)\|_{L^2(\mathbb{R})} \\
 & \leq c(T) \left( \Delta(0) + \int_0^T \|\square(\phi(s, \cdot) - \phi'(s, \cdot))\|_{L^2(\mathbb{R})} ds \right).
 \end{aligned} \tag{22}$$

In view of Eq. (15), in order to estimate the last term above, we have to estimate four terms involving the null forms  $\mathcal{Q}$ . We give the estimate of one of the terms only, the rest of them being similar,

$$\begin{aligned}
 & \int_0^T \|Q_0(\Psi'^\dagger - \Psi^\dagger, \gamma^0 \Psi')\|_{L^2(\mathbb{R})} ds \leq T^{1/2} \|Q_0(\Psi'^\dagger - \Psi^\dagger, \gamma^0 \Psi')\|_{L^2(\mathbb{R} \times \mathbb{R})} \\
 & \leq c \left( \Delta(0) + \int_0^T \|\square(\Psi - \Psi')\|_{L^2(\mathbb{R})} ds \right) \\
 & \quad \times \left( J'(0) + \int_0^T \|\square \Psi'\|_{L^2(\mathbb{R})} ds \right),
 \end{aligned} \tag{23}$$

where we have used Lemma 1. Using (20) we get

$$\begin{aligned} \int_0^T \|\square(\Psi - \Psi')(s, \cdot)\|_{L^2(\mathbb{R})} ds &= \int_0^T \|\mathcal{D}(\psi - \psi')\|_{L^2(\mathbb{R})} ds \\ &\leq c(1 + J(0)) T \Delta(T). \end{aligned}$$

Similarly

$$\int_0^T \|\square\Psi'(s, \cdot)\|_{L^2(\mathbb{R})} ds \leq C(T, J'(0)).$$

Hence

$$\int_0^T \|Q_0(\Psi'^\dagger - \Psi^\dagger, \gamma^0\Psi')\|_{L^2(\mathbb{R})} ds \leq C(T, J(0), J'(0))(\Delta(0) + T \Delta(T)). \quad (24)$$

Thus (22) gives

$$\begin{aligned} &\|\phi(t, \cdot) - \phi'(t, \cdot)\|_{H^1(\mathbb{R})} + \|\partial_t \phi(t, \cdot) - \partial_t \phi'(t, \cdot)\|_{L^2(\mathbb{R})} \\ &\leq C(T, J(0), J'(0))(\Delta(0) + T \Delta(T)). \end{aligned} \quad (25)$$

Since at this point in our proof we are only interested in proving local existence we may assume that  $T \leq 1$ . Then

$$C(T, J(0), J'(0)) \leq C(1, J(0), J'(0)) := C(J(0), J'(0)).$$

From (21) and (25) we get

$$\Delta(T) \leq C(J(0), J'(0))(\Delta(0) + T \Delta(T)).$$

A simple bootstrap argument gives Proposition 1.

### 3. THE GLOBAL PROBLEM

The local solution constructed above can actually be extended globally in time. Recall that in Lemma 4 we proved that given any time interval  $[0, T)$ , with  $T$  not necessarily small, and for any charge class solution defined on  $[0, T) \times \mathbb{R}$  we have  $J(T) < +\infty$ . One then proceeds exactly as in the last section of 3 to extend the solution to all times. This type of argument is actually a variant of Segal's original ideas [5] adapted to the case of low regularity solutions.

## 4. REMARK

The choice of spaces in the local existence theorem of Subsection 2.2 was dictated by our desire to prove the global existence of solutions of (1)–(2). However, if one is only interested in local solutions then Theorem 2 can be significantly improved. Using space time estimates for null forms we can prove local existence with  $\psi \in L^2$  and  $\phi \in H^{1/2}$ . However, based on scaling arguments, we conjecture that the Cauchy problem for the Dirac–Klein–Gordon equations is well posed in the space  $\psi \in H^{-1/2}$  and  $\phi \in L^2$ . The proofs will appear elsewhere.

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